Dynamical r-matrices for Hitchin's systems on Schottky curves

B. Enriquez

Abstract. *We express Hitchin's systems on curves in Schottky parametrization, and construct dynamical r-matrices attached to them.*

Introduction.

This paper is devoted to the study of Hitchin's integrable systems. These systems are defined on the cotangent space to the moduli space of holomorphic G-bundles on a Riemann surface. One source of the recent interest in these systems is that they can be viewed as a classical limit of the $\mathcal{D}$-modules proposed by Beilinson and Drinfeld in their approach to the geometric Langlands correspondence ([BD]).

A classical approach to integrable systems is the r-matrix approach; this viewpoint usually leads to their quantization. In a previous work ([ER]), we constructed such r-matrices in the case of Hitchin systems in genus one. These r-matrices are dynamical; it means that they depend on a (Poisson commutative) set of phase space variables. Due to this dependence, they satisfy a dynamical generalization of the classical Yang-Baxter equation (DYBE). In his work [F], G. Felder derived the same r-matrices from considerations related to the Knizhnik-Zamolodchikov-Bernard equation. Moreover, he constructed a quantum version of the generalized Yang-Baxter equation and quantum groups attached to them.

In this paper, we formulate Hitchin's integrable systems on a higher genus Riemann surface, uniformized à la Schottky, and point out a connection with the Garnier systems in the case of Mumford curves. (Such a formulation can also be found in [O].) We then derive a dynamical r-matrix for these systems. This gives an alternative proof of their integrability. We also show that they satisfy an analogue of the DYBE, but it seems difficult to formulate correctly a quantum counterpart of this equation.

Let us also note here that an adelic analogue of the r-matrix presented here has been found by G. Felder in [Fe]. It would be interesting to relate both constructions. We also hope that the version presented here of the DYBE may help to understand the correct "higher genus" version of the formalism of P. Etingof and A. Varchenko ([EV]).

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1. Hitchin's systems with level structures.

We recall this notion from [Be], [Ma]. Let X be a compact complex curve, $D = \sum_i [P_i]$ a divisor on X (all points P_i are assumed different). Let G be a complex reductive group, $\mathbf{g}$ be its Lie algebra. Let $\mathcal{M}_{G,D}(X)$ the moduli space of pairs (P, j_i) of a principal G-bundle over X and of trivialisations of P above each point P_i. Then the cotangent space to $\mathcal{M}_{G,D}(X)$ at (P, j_i) is identified with $H^0(X, \mathbf{g}_P \otimes \Omega^1(-D))$, where $\mathbf{g}_P = P \times_G \mathbf{g}$ ($\mathbf{g}$ being viewed as the adjoint representation of G), and Ω^1 being the canonical bundle of X. We have a natural mapping ([Hi]) $T^*\mathcal{M}_{G,D}(X) \to \oplus_{i=1}^r H^0(X, \Omega^1(-D)^{\otimes d_i})$, r being the rank of G and d_i the exponents of G ($d_i = \deg P_i$, for $(P_i)_{1 \le i \le r}$ a basis of the invariant polynomials on $\mathbf{g}$), obtained by applying P_i on $H^0(X, g_P \otimes \Omega^1(-D))$. Functions on $T^*\mathcal{M}_{G,D}(X)$ obtained from this mapping are in involution, $T^*\mathcal{M}_{G,D}(X)$ being endowed with its natural symplectic structure.

2. Schottky uniformisation.

Let $l \ge 1$ be an integer, and let on $\mathbf{C}P^1$, Γ_i and Γ'_i ($i = 1, \cdots, l$) be $2l$ circles, bounding $2l$ disjoint open discs D_i and D'_i. Let us give ourselves l elements γ_i of $SL(2, \mathbf{C})$ mapping Γ_i to Γ'_i; and let X be the Riemann surface

$$X = \mathbf{C}P^1 - \bigcup_{i=1}^{l} D_i \cup D'_i / (x_i \sim \gamma_i(x_i), x_i \in \Gamma_i).$$

An open subset of the moduli space $\mathcal{M}_G(X)$ of principal G-bundles over X can be identified with G^l/G (where G acts on each factor of G^l by the adjoint action), by associating to the class of $(g_i)_{1 \le i \le l}$ the class of the bundle $P_{(g_i)} = \big[\mathbf{C}P^1 - \bigcup_{i=1}^{l} D_i \cup D'_i\big] \times G / ((x_i, g) \sim (\gamma_i(x_i), g_i g), x_i \in \Gamma_i, g \in G)$.

Assume for simplicity, that ∞ belongs to $\mathbf{C}P^1 - \bigcup_{i=1}^{l} D_i \cup D'_i$. Then an open subset of $\mathcal{M}_{G,[\infty]}(X)$ can be identified with G^l: adjoin to the bundle defined above, the identity mapping from the fibre at ∞ to G.

Let us identify now the cotangent bundle to G with $G \times \mathbf{g}^*$ via left invariant one-forms, and accordingly T^*G^l with $G^l \times \mathbf{g}^{*l}$. Elements of $H^0(X, \mathbf{g}_P \otimes \Omega^1(-[\infty]))$ are given by twisted holomorphic one-forms, with poles at ∞, that is by one-forms over $\mathbf{C}P^1 - \Lambda$ (Λ is the limit set of the free group Γ generated by the γ_i) $\omega(z)dz$, such that

$$\omega(\gamma z)\gamma'(z)dz = \mathrm{Ad}(g_\gamma)\omega(z)dz, \qquad (1)$$

for all $\gamma \in \Gamma$ and with only poles at $\Gamma\infty$. Recall the definition of the multiplier q_γ of an hyperbolic element γ of $SL(2, \mathbf{C})$: the transformation defined by γ verifies $\frac{\gamma(z) - a_\gamma}{\gamma(z) - b_\gamma} = q_\gamma \frac{z - a_\gamma}{z - b_\gamma}$, for certain a_γ and b_γ in $\mathbf{C}P^1$ and $q_\gamma \in \mathbf{C}^\times$, $|q_\gamma| < 1$. We have:

Lemma 1.— *Over the open subset of G^l defined by*

$$\sum_{i=1}^{l} q_{\gamma_i}(\|\mathrm{Ad} g_i\| + \|\mathrm{Ad} g_i^{-1}\|) < 1,$$

($\| \ \|$ an algebra norm in End g), the twisted holomorphic one-form corresponding to $(g_i, \xi_i) \in G^l \times \mathbf{g}^{*l}$ is given by the Poincaré series (cf. also [B])

$$\xi(g_i, \xi_i)(z)dz = \sum_{\gamma \in \Gamma, 1 \le i \le l} \frac{\mathrm{Ad}(g_\gamma^{-1})\xi_i}{\gamma(z) - \gamma_i^{-1}(\infty)} \gamma'(z)dz. \tag{2}$$

Proof. Let us prove the convergence of (2) under the present hypothesis (see also [Bu], [Fo]). Let $\gamma = \gamma_{i_1}^{\epsilon_1} ... \gamma_{i_p}^{\epsilon_p}$, $i_k = 1, ..., l$, $\epsilon_{i_k} = \pm 1$. The norm of $\gamma'(z)$ is then bounded by $Cq_{i_1}...q_{i_p}$ (C a fixed constant). On the other hand, the norm of $\mathrm{Ad}(g_\gamma)\xi_i$ is estimated by

$$\|\mathrm{Ad}(g_{i_1}^{\epsilon_1})\| ... \|\mathrm{Ad}(g_{i_p}^{\epsilon_p})\| \|\xi_i\|;$$

it follows that the contribution to (2) of elements of Γ, of length equal to p, is bounded by

$$C\Big(\sum_{i=1}^l q_i(\|\mathrm{Ad}g_i\| + \|\mathrm{Ad}g_i^{-1}\|)\Big)^p (\sum_{i=1}^l \|\xi_i\|);$$

and the sum of all these terms is bounded by

$$(1 - \sum_{i=1}^l q_i(\|\mathrm{Ad}g_i\| + \|\mathrm{Ad}g_i^{-1}\|))^{-1} (\sum_{i=1}^l \|\xi_i\|).$$

We prove similarly that the sum (2) is an analytic function of z.

Let $(X_i)_{1 \le i \le l}$ be l elements of the Lie algebra of G, and let us view them as infinitesimal left-invariant translations. The pairing between (ξ_i) and (X_i) is simply given by $\langle (\xi_i), (X_i) \rangle = \sum_{i=1}^l \langle \xi_i, X_i \rangle$. On the other hand, the pairing between $\xi(z)dz$ and (X_i) is given by $\sum_{i=1}^l \frac{1}{2i\pi} \int_{\Gamma_i} \langle X_i, \xi(z)dz \rangle$. Recall that the poles of the one-form $\frac{\gamma'(z)dz}{\gamma(z) - \gamma_i^{-1}(\infty)}$ are located at $\gamma^{-1}(\infty)$ and $\gamma^{-1}\gamma_i^{-1}(\infty)$. The only possibility for them to be on two different sides of Γ_i is $\gamma = e$. The contribution of the corresponding term is, by the residues formula, $\langle \xi_i, X_i \rangle$. ∎

For the rest of the paper, we will work in the open subset defined in prop. 1.

Remarks.

1) The present formalism can be adapted to the situation of Mumford curves, where the q_{γ_i} are considered as formal variables. The base ring is then $R = \mathbf{C}[[q_{\gamma_1}, \cdots, q_{\gamma_l}]]$. If the g_{γ_i} belong to $G(R)$, and the ξ_i to $\mathbf{g} \otimes R$, the series (2) converges without restrictions; it then has to be interpreted as a formal series of the type $dz \sum_{1 \le i \le l, n \ge 1} \frac{\alpha_{i,n}}{(z - a_{\gamma_i})^n} + \frac{\beta_{i,n}}{(z - b_{\gamma_i})^n}$, $\alpha_{i,n}$, $\beta_{i,n}$ in $\mathbf{g} \otimes R$ and with valuation $\ge n$, subject to the conditions $\sum_{i=1}^l a_{\gamma_i}\alpha_{i,1} + b_{\gamma_i}\beta_{i,1} = 0$ of regularity at ∞. Note that Lax operators of this type (with bounded orders of poles) already appeared in the work of Garnier [Ga].

2) We cannot use $\frac{\gamma'(z)dz}{\gamma(z) - \gamma_i^{-1}(\infty)} = \frac{dz}{z - (\gamma_i\gamma)^{-1}(\infty)} - \frac{dz}{z - \gamma^{-1}(\infty)}$ to regroup terms in (2) and obtain for expression of the series (2), $\sum_{\gamma \in \Gamma} \mathrm{Ad}g_\gamma^{-1}(\sum_{i=1}^l \mathrm{Ad}g_{\gamma_i}\xi_i - \xi_i) \frac{dz}{z - \gamma^{-1}(\infty)}$,

since the last expression does not converge; but we can deduce from it the variation of (2) under replacement of ∞ by a point z_0 close to it. The proper replacement of (2) is then $\sum_{\gamma\in\Gamma,1\le i\le l}\mathrm{Ad}(g_\gamma^{-1})\xi_i(\frac{\gamma'(z)}{\gamma(z)-\gamma_i^{-1}(z_0)}-\frac{\gamma'(z)}{\gamma(z)-z_0})dz$, whose derivative w.r.t. z_0 is $\sum_{\gamma\in\Gamma}\mathrm{Ad}g_\gamma^{-1}(\sum_{i=1}^l\left(\mathrm{Ad}g_{\gamma_i}\xi_i-\xi_i\right))\frac{(\gamma^{-1})'(z_0)dz}{(z-\gamma^{-1}(z_0))^2}$. In particular, this variation is zero under the condition $\sum_{i=1}^l(\mathrm{Ad}g_{\gamma_i}\xi_i-\xi_i)=0$, which is also the condition for (2) to be regular at ∞ (and also the condition that the image of (g_i,ξ_i) by the moment map associated to the adjoint action of G on G^l is zero).

3. Dynamical r-matrices.

Consider the "r-matrices"

(3)

$$r(z,w)dz=\sum_{\gamma\in\Gamma}\frac{\mathrm{Ad}g_\gamma^{(2)}P}{\gamma(z)-w}\gamma'(z)dz,\quad s(z,w)dw=\sum_{\gamma\in\Gamma}\frac{\mathrm{Ad}g_\gamma^{(1)}P}{z-\gamma(w)}\gamma'(w)dw$$
$$=-r(w,z)^{(21)}dw,$$

where P is the split Casimir element of $\mathbf{g}\otimes\mathbf{g}$ (differential elements dz and dw will be considered to commute together). The proof of convergence of analyticity of these series is similar to that of the series (2).

We wish to prove:

Proposition.— *The Poisson brackets of operators ξ are given by*

(4)

$$\{\xi(g_i,\xi_i)(z)^{(1)}dz,\xi(g_i,\xi_i)(w)^{(2)}dw\}=[r(z,w)dz,\xi(g_i,\xi_i)(w)^{(2)}dw]$$
$$+[s(z,w)dw,\xi(g_i,\xi_i)(z)^{(1)}dz].$$

For this we first compare the transformation properties of both sides of (4), under the action of γ_i on z. Call the l.h.s. and r.h.s. of (4) respectively $A(z,w)dzdw$ and $B(z,w)dzdw$, and set $C(z,w)dzdw=A(z,w)dzdw-B(z,w)dzdw$. We have:

Lemma 2.— *$C(z,w)dzdw$ satisfies*

$$C(\gamma z,w)\gamma'(z)dzdw=\mathrm{Ad}g_\gamma^{(1)}C(z,w)dzdw,\tag{5}$$

$$C(z,\gamma w)\gamma'(w)dzdw=\mathrm{Ad}g_\gamma^{(2)}C(z,w)dzdw,\tag{6}$$

$$C(w,z)dzdw=-C(z,w)^{(21)}dzdw;\tag{7}$$

moreover, $C(z,w)dzdw$ has no poles on $(\mathbf{C}P^1-\Lambda)^2$.

Proof. We have

$$\begin{aligned}A(\gamma_i z, w)\gamma_i'(z)dzdw &= \mathrm{Ad}g_i^{(1)} A(z,w)dzdw \\ &\quad + [\{g_i^{(1)}, \xi(w)^{(2)}dw\} g_i^{-1(1)}, \mathrm{Ad}g_i^{(1)}\xi(z)^{(1)}dz],\end{aligned}$$

and since

$$r(\gamma_i z, w)\gamma_i'(z)dz = \mathrm{Ad}g_i^{(1)} r(z,w)dz, \quad \begin{aligned} s(\gamma_i z, w)dw =& \mathrm{Ad}g_i^{(1)} s(z,w)dw \\ &+ \sum_{\gamma\in\Gamma} \frac{\mathrm{Ad}g_{\gamma_i\gamma}^{(1)} P\gamma'(w)dw}{\gamma(w)-\gamma_i^{-1}(\infty)}\end{aligned}$$

(the second identity is obtained using $d_w \ln(\gamma_i\gamma(w) - \gamma_i(z)) - d_w \ln(\gamma(w) - z) = d_w \ln(\gamma(w) - \gamma_i^{-1}(\infty))$), we obtain

$$\begin{aligned}B(\gamma_i z, w)\gamma_i'(z)dzdw =& \mathrm{Ad}g_i^{(1)} B(z,w)dzdw \\ &+ [\sum_{\gamma\in\Gamma} \frac{\mathrm{Ad}g_{\gamma_i\gamma}^{(1)} P}{\gamma(w)-\gamma_i^{-1}(\infty)}\gamma'(w)dw, \mathrm{Ad}g_i^{(1)}\xi(z)^{(1)}dz].\end{aligned}$$

We then compute $\{g_i^{(1)}, \xi(w)^{(2)}dw\} g_i^{-1(1)}$, and find using $\{g_i^{(1)}, \xi_j^{(2)}\} = \delta_{ij} g_i^{(1)} P$ that it is equal to $\sum_{\gamma\in\Gamma} \frac{\mathrm{Ad}g_{\gamma_i\gamma}^{(1)} P}{\gamma(w)-\gamma_i^{-1}(\infty)}\gamma'(w)dw$. This proves (5). (7) is clear, and together with (5) it implies (6).

Let us turn to the statement about poles. Let us fix $w \notin \Gamma\infty$. Then $A(z,w)dzdw$ has poles when $z \in \Gamma\infty$. We have for z near ∞, $\xi(z)dz = \sum_{i=1}^{l} (\mathrm{Ad}g_i \xi_i - \xi_i)\frac{dz_\infty}{z_\infty}$+reg., $z_\infty = 1/z$ is a local coordinate near ∞. Then we have

$$\{\sum_{i=1}^{l} (\mathrm{Ad}(g_i)\xi_i - \xi_i)^{(1)}, \xi(w)^{(2)}dw\} = -[P, \xi(w)^{(2)}dw],$$

so that $A(z,w)dzdw = -[P, \xi(w)^{(2)}dw]\frac{dz_\infty}{z_\infty}$+reg. The expansion of the first part of $B(z,w)dzdw$ is the same since $r(z,w)dz = -P\frac{dz_\infty}{z_\infty}$+reg. near $z=\infty$. The second part of $B(z,w)dzdw$ is regular since $\xi(z)dz$ has a pole of order one, whereas $s(z,w)dw$ tends to zero as $z \to \infty$. So $C(z,w)dzdw$ has no poles for $z\to\infty$; it has no poles either for $w \to \infty$ because of (7), and these are all the poles of $A(z,w)dzdw$. The poles of $B(z,w)dzdw$ are the same, with the possible addition of $z \in \Gamma w$. Because of (5), we can restrict ourselves to the study of the pole at $z = w$; but this pole does not occur because of the g-invariance of P. ■

We now show:

Lemma 3.— *$C(z,w)dzdw = 0$, so that (4) is valid.*

Proof. Due to our assumptions on (g_i), we know that any twisted (by (g_i)) holomorphic one-form on X, with possible pole at $[\infty]$ can be written in the form of a Poincaré series (since these series converge, and the dimension of the vector space they form is equal to the dimension of $H^0(X, P_{(g_i)} \times_G \mathbf{g}(-[\infty]))$ as it can be computed using the Riemann-Roch formula). It follows that there exist elements C_{ij} of $\mathbf{g} \otimes \mathbf{g}$, such that $C(z,w)dzdw = \sum_{1 \le i,j \le l, \gamma, \delta \in \Gamma} \frac{\gamma'(z)dz}{\gamma(z) - \gamma_i^{-1}(\infty)} \frac{\delta'(w)dw}{\delta(w) - \gamma_j^{-1}(\infty)} \mathrm{Ad} g_\gamma^{(1)} \mathrm{Ad} d_\gamma^{(2)} C_{ij}$. C_{ij} can be computed by

$$C_{ij} = \int_{\Gamma_i \times \Gamma_j} C(z,w)dzdw.$$

Now, we have $\int_{\Gamma_i \times \Gamma_j} A(z,w)dzdw = \{\xi_i^{(1)}, \xi_j^{(2)}\} = \delta_{ij}[P, \xi_i^{(1)}]$, and

$$\int_{\Gamma_i \times \Gamma_j} B(z,w)dzdw = \int_{\Gamma_i \times \Gamma_j} [r(z,w)dz, \xi(g_i,\xi_i)(w)^{(2)}dw] + \int_{\Gamma_i \times \Gamma_j} [s(z,w)dw, \xi(g_i,\xi_i)(z)^{(1)}dz],$$

for $i \neq j$. In $\int_{\Gamma_i \times \Gamma_j} [r(z,w)dz, \xi(g_i,\xi_i)(w)^{(2)}dw]$, we integrate first w.r.t. z; expanding $r(z,w)dz$ according to (3), we have to integrate the one-form $d_z \ln(\gamma(z) - w)$; it has poles at $\gamma^{-1}(w)$ and $\gamma^{-1}(\infty)$; w being on Γ_j, these two points are always on one and the same side of Γ_i, so that this term does not contribute to the integral. Exchanging the roles of z and w, we find that $\int_{\Gamma_i \times \Gamma_j} [s(z,w)dw, \xi(g_i,\xi_i)(z)^{(1)}dz]$ is also equal to zero. Finally, $C_{ij} = 0$ for $i \neq j$.

For $i = j$, we consider a deformation Γ_i^ϵ of Γ_i, encircling Γ_i and within the domain $\mathbf{C}P^1 - \bigcup_{i=1}^l D_i \cup D_i'$. We have still, $C_{ii} = \int_{\Gamma_i \times \Gamma_i^\epsilon} C(z,w)dzdw$, and

$$\int_{\Gamma_i \times \Gamma_i^\epsilon} A(z,w)dzdw = [P, \xi_i^{(1)}].$$

Now,

$$\int_{\Gamma_i \times \Gamma_i^\epsilon} B(z,w)dzdw = \int_{\Gamma_i \times \Gamma_i^\epsilon} [r(z,w)dz, \xi(g_i,\xi_i)(w)^{(2)}dw] + \int_{\Gamma_i \times \Gamma_i^\epsilon} [s(z,w)dw, \xi(g_i,\xi_i)(z)^{(1)}dz]. \tag{8}$$

Repeating the reasoning above, and due to the relative configurations of Γ_i and Γ_i^ϵ, the first term of the r.h.s. of (8) is again zero. For the second term, we have

$$\int_{\Gamma_i \times \Gamma_i^\epsilon} [s(z,w)dw, \xi(g_i,\xi_i)(z)^{(1)}dz] = \sum_{\gamma \in \Gamma} \int_{\Gamma_i \times \Gamma_i^\epsilon} [\frac{\mathrm{Ad} g_\gamma^{(1)} P}{z - \gamma(w)} \gamma'(w)dw, \xi(z)^{(1)}dz]$$

and the only non zero contribution is from the term with $\gamma = 1$; it gives

$$\int_{\Gamma_i} dz[\int_{\Gamma_i^\epsilon} \frac{Pdw}{z-w}, \xi(z)^{(1)}] = \int_{\Gamma_i} dz[P, \xi(z)^{(1)}] = [P, \xi_i^{(1)}].$$

So $C_{ii} = 0$. ■

Remarks.

1. It is interesting to give a purely algebraic meaning to the r-matrices of (3). Setting $\rho_{\gamma_i} = \sum_{\gamma\in\Gamma} \frac{\mathrm{Ad}(g_\gamma^{-1} g_{\gamma_i}^{-1})^{(2)} P}{\gamma(w)-\gamma_i^{-1}(\infty)} \gamma'(w) dw \in \mathbf{g} \otimes H^0(X, \Omega^1 \otimes \mathbf{g}_P(-[\infty]))$ and defining ρ_γ, for $\gamma \in \Gamma$ by the cocycle condition $\rho_{\gamma\gamma'} = \mathrm{Ad} g_{\gamma'}^{(1)} \rho_\gamma + \rho_{\gamma'}$, we have constructed a 1-cocycle $\rho \in H^1(X, g_P \otimes H^0(X, \Omega^1 \otimes g_P(-[\infty]))) = H^1(X, \mathbf{g}_P) \otimes H^0(X, \Omega^1 \otimes \mathbf{g}_P(-[\infty]))$. (By Serre duality we have a natural element in $H^1(X, \mathbf{g}_P) \otimes H^0(X, \Omega^1 \otimes \mathbf{g}_P)$ but ρ seems a little different.)

ρ then serves to define an affine spaces bundle over the vector bundle $H^0(X, \Omega^1 \otimes \mathbf{g}_P(-[\infty])) \otimes \mathbf{g}_P$ over X, and also over the bundle with fiber at $x \in X$, $H^0(X, \Omega^1 \otimes \mathbf{g}_P(-[\infty] - [x])) \otimes \mathbf{g}_P$; r can then be viewed as a section of the twist of this last bundle. Probably, there is a natural twist of $(\Omega^1 \otimes \mathbf{g}_P) \boxtimes \mathbf{g}_P(-[\infty])$ over $X \times X$, for which r could be considered as a section.

2. The result of [BV] states the local existence of an r-matrix for general integrable systems; but it is easy to see that the r-matrices constructed in the situation of Hitchin system, using the methods of this work, would depend on (ξ_i) (and in fact not be defined for $\xi_i = 0$).

4. Jacobi identity

Writing that (4) satisfies the Jacobi identity, we find the dynamical Yang-Baxter equation:

$$\begin{aligned}
&[r^{32}dz_3, r^{21}dz_2] + [r^{23}dz_2, r^{31}dz_3] + [r^{31}dz_3, r^{21}dz_2] + \\
&\sum_{\gamma\in\Gamma, 1\le i\le l} \frac{\mathrm{Ad} g_\gamma^{-1}(e^{\alpha(3)})\gamma'(z_3)dz_3}{\gamma(z_3) - \gamma_i^{-1}(\infty)} \partial_{e_\alpha^{(i)}}(r^{21}dz_2) \\
&+ \sum_{\gamma\in\Gamma, 1\le i\le l} \frac{\mathrm{Ad} g_\gamma^{-1}(e^{\alpha(2)})\gamma'(z_2)dz_2}{\gamma(z_2) - \gamma_i^{-1}(\infty)} \partial_{e_\alpha^{(i)}}(r^{31}dz_3) = 0,
\end{aligned} \tag{9}$$

where $P = e^\alpha \otimes e_\alpha$, $\partial_{x^{(i)}}$ is the vector field on G^l, given by the left translation by $x \in g$ on the i-th factor and zero on the other factors.

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B.E.: Centre de Mathématiques, URA 169 du CNRS, Ecole Polytechnique, 91128 Palaiseau, France